Int. J. Solids Structures Vol. 14, pp. 499–516
0020-7683/78/0601-0499/$02.00/0

GENERAL METHOD FOR ANALYSING CONTACT STRESSES ON CYLINDRICAL VESSELS

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(*Received 7 March 1977; in revised form 28 November 1977; received for publication 3 January 1978*)

Abstract—A general theory is presented for the prediction of the stresses, displacements and interface pressure of cylindrical vessels partially filled with liquid, pressurized and having surfaces of contact with rigid supports. The supports are subdivided into a number of line and surface elements in the axial and circumferential directions. Each element is subjected to a load q_i, $i = 1, 2, \ldots, N$. The applied loads, expressed in double Fourier series and inserted into shells equations of motion, allow the determination of the corresponding displacements and stresses in terms of the q_i's. Prediction of the saddle-vessel interface pressure distribution is obtained by minimizing the potential energy of the system. Some calculations are conducted to illustrate the theory. The theory is compared with experiment, and agreement is found to be quite good.

INTRODUCTION

Knowledge of the pressure distribution of the surface of contact between fluid-filled cylindrical vessels and its supports is of considerable practical interest, considering that most cylindrical vessels are utilized in containing or conveying fluids. Although the problem of determining the stresses and displacements of saddle supported cylindrical vessels using beam's theory or semi-empirical approach has produced many papers[1–4], the literature reveals a limited number of rigorous analysis which have been generally developed using cylindrical shells' theory.

For instance, Forbes and Tooth[5], and Wilson and Tooth[6] formulate an analysis capable of predicting the saddle/cylinder interface pressure and the stress resultant throughout the vessel. The most severe limitation of their theory is the assumption, in their investigation, of constant interface pressure distribution in the axial direction. The authors of this paper believe that the interface pressure is non-uniform in the axial as well as in the circumferential directions. The need is evident for a theory which can be used for the static or dynamic analysis of any kind of circular cylindrical shells partially filled with liquid, pressurized and having surfaces of contact with rigid supports. This theory may be used advantageously in the stress analysis of heat exchanger for nuclear power plants.

The work presented here based on a recently developed theory[7] by the authors, is an attempt to produce a general theory for the prediction of the saddle-vessel interface pressure distribution in the axial and circumferential directions as well as the stresses and displacements at any point within the structure with wide range of applicability.

To this end, the support is subdivided into a sufficient number of line and surface elements each of which subjected to a load q_i, $i = 1, 2, \ldots, N$. Prediction of the saddle-vessel interface pressure distribution is obtained by minimizing the total Potential energy of the system which is derived in terms of the q_i's. A number of assumptions are made during the course of the investigation; a compendium of these assumptions and the limitations of the theory will be given in the text.

The theory of Ref. [7], and hence the theory of this paper, is capable of analysing geometrically axially non-symmetric, long or short, thin cylindrical shells subject to any set of boundary conditions (including supports other than at the two axial extremities of the shell).

The organization of the paper is as follows. First, the matrix formulation of the problem is presented and the field pressure distribution of the saddle/cylinder is transformed to a discrete set of forces. Secondly, the stresses and displacements of the shell are expressed in terms of these forces q_i, $i = 1, 2, \ldots, N$. Then, the potential energy of the system is presented, and its minimization is obtained. Finally, the method of calculation is developed and some results, conducted to illustrate the theory, are discussed.

2. MODELING OF THE SUPPORT

The basic equations which describe the static behaviour of cylindrical shells with bending resistance under arbitrary loads are determined from Sanders' equations of equilibrium of thin shells[8, 9]. This shell theory was preferred because all strains vanish for small rigid body motions. This matter is further elaborated in Ref. [9], where the equilibrium equations are given in terms of the axial, circumferential and radial displacements of the middle surface, U, V and W, respectively (Fig. 1). The stress–strain and the strain-displacements relations are also listed in Ref. [9].

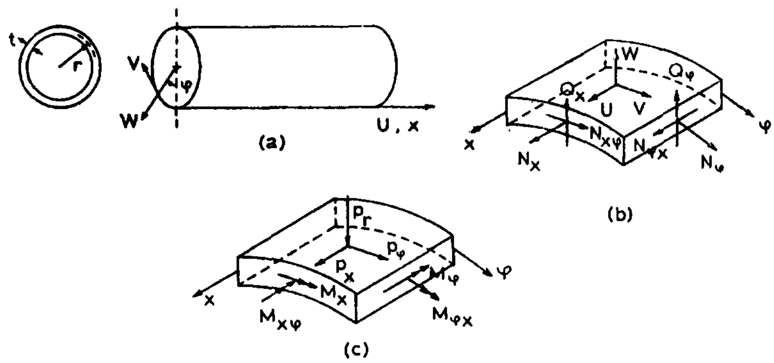

Fig. 1. (a) Definition of the displacements U, V and W. (b) Stress-resultants and displacements acting upon a differential elements. (c) Stress couples and surface loads acting upon a differential elements.

A given cylindrical shell partially-filled with liquid, pressurized and having a surface of contact with a rectangular rigid support of dimensions $(a, 2b)$ is shown in Fig. 2. The locations of the support on the vessel is arbitrary, with coordinates δ_o and (x_o, x_f) in the circumferential and axial directions, respectively. The theory developed here for one support may be applied to shells having two or more supports. Also the assumption of a vertical plane of symmetry of the loads through the axis of the shell, permits the investigation of only half the support (a, b).

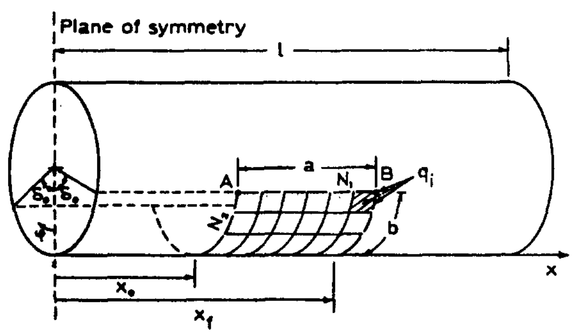

Fig. 2. Location of the support on the shell. (N_1 and N_2 are the number of line elements at the boundaries of the support in x and φ directions, respectively; and, $N = N_1N_2 + 2N_2 + N_1 + 2$, is the total number of elements).

These dimensions "a" and "b" are first subdivided into N_1 and N_2 line elements, respectively; then by assuming two point loads on the boundaries A and B of the support (Fig. 2), N finite elements are distributed over the area (a, b) as follows: (1) two concentrated loads of densities q_i (lb or kg), $i = 1, 2$, applied on the points A and B; (2) $(N_1 + N_2)$ line loads applied on the line elements of densities q_i (lb/in or kg/m), $i = 3, 4, \ldots, 2N_2 + N_1 + 2$; and (3) N_1N_2 surface loads on densities q_i (lb/in^2 or kg/m^2), $i = 2N_2 + N_1 + 3, \ldots, N$; where N is given by

$$N = N_1N_2 + 2N_2 + N_1 + 2. \tag{1}$$

The pressure distribution on the possible surface of contact between the rigid support and the shell is represented by the densities q_i's where $i = 1, 2, \ldots N$.

3. STRESSES AND DISPLACEMENTS DUE TO ARBITRARY LOADING

In this section we shall develop a general procedure in order to obtain the displacements and the stresses induced by the q_i's, fluid surcharge pressure, weight of the vessel and its heads. To do so, the loads and the displacements are first represented by double Fourier series and then inserting these series into the equations of motion, the coefficients of the Fourier series are obtained for the displacement components in terms of the load factors p_{xmn}, $P_{\phi mn}$ and P_{rmn} for the axial, circumferential and radial directions, respectively. The second step is to determine these loads factors P_{xmn}, $P_{\phi mn}$ and P_{rmn} in terms of the unknowns q_i's for each particular case.

3.1 *Arbitrary loading and load factors*

Using the method of Fourier expansions and assuming that, for a cylinder of length l,

$$\begin{Bmatrix} p_x \\ p_r \\ p_\phi \end{Bmatrix} = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} [T_n][T_{mx}] \begin{Bmatrix} p_{xmn} \\ p_{rmn} \\ p_{\phi mn} \end{Bmatrix}, \tag{2}$$

and

$$\begin{Bmatrix} U_p \\ W_p \\ V_p \end{Bmatrix} = \sum_{m=1}^{\infty} \sum_{n=0}^{\infty} [T_n][T_{mx}] \begin{Bmatrix} u_{mn} \\ w_{mn} \\ v_{mn} \end{Bmatrix}, \tag{3}$$

it is possible to obtain, by introducing (2) and (3) into eqn (38) of Ref. [9], the coefficient of the Fourier series for the displacement components as follows:

$$\begin{Bmatrix} u_{mn} \\ w_{mn} \\ v_{mn} \end{Bmatrix} = \frac{r^2}{D} [A_F]^{-1} \begin{Bmatrix} p_{xmn} \\ -p_{rmn} \\ p_{\phi mn} \end{Bmatrix}, \tag{4}$$

where m and n are the axial and circumferential wave number, respectively, and the matrices $[T_n]$, $[T_{mx}]$ and $[A_F]$ are shown in the Appendix.

Equations (2)–(4) allow the loads and displacements to be expressed in terms of the load factors P_{xmn}, P_{rmn} and $P_{\phi mn}$. The object now is to determine the load factors corresponding to each particular case of loading.

(a) *Concentrated radial loads.* We assume two concentrated radial forces of intensity q_i (lb or kg) applied at the coordinate $(x, \phi) = (b_i, \pm\delta_i)$ (Fig. 3).

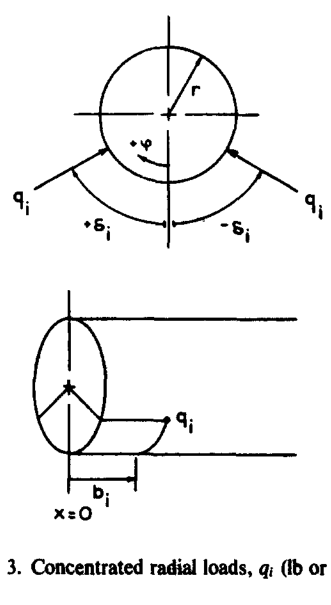

Fig. 3. Concentrated radial loads, q_i (lb or kg).

The load factors corresponding to such forces are as follows

$$p_{rmoi} = \frac{2q_i}{\pi r l} \sin \frac{m\pi b_i}{l}, \qquad m = 1, 2, 3, \ldots, \tag{5a}$$

and

$$p_{rmni} = \frac{4q_i}{\pi r l} \cos n\delta_i \sin \frac{m\pi b_i}{l}, \qquad n, m = 1, 2, 3, \ldots, \tag{5b}$$

where $i = 1$ and 2. By referring to Fig. 2 we obtain $\delta_1 = \delta_2 = \delta_o$, $b_1 = x_o$ and $b_2 = x_f$.

(b) *Line loads on two segments along the generator.* Let the segments, on which a constant inward line load q_i (lb/in or kg/m) is applied, be centered at $(x, \phi) = (b_i, \pm\delta_i)$ and of dimension $2c_2$ along the axial direction (Fig. 4).

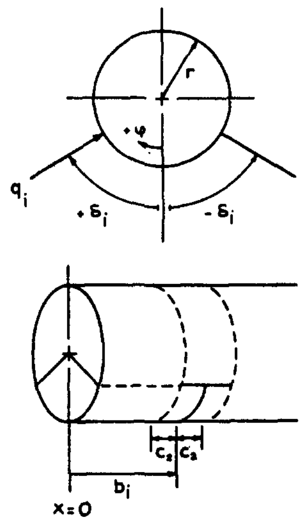

Fig. 4. Line load along a generator, q_i (lb/in. or kg/m).

The coefficients of the series expansion for this loading is given by

$$p_{rmoi} = \frac{4q_i}{\pi^2 rm} \sin \frac{m\pi C_2}{l} \sin \frac{m\pi b_i}{l}, \qquad m = 1, 2, 3, \ldots,$$

and

$$p_{rmni} = \frac{8q_i}{\pi^2 rm} \sin \frac{m\pi C_2}{l} \cos n\delta_i \sin \frac{m\pi b_i}{l}, \qquad n, m = 1, 2, 3, \ldots, \tag{6b}$$

where $i = 3, 4, \ldots, N_1 + 2$.

(c) *Line loads on two segements perpendicular to the generator.* For the cases where the shell is subjected to a line load q_i (lb/in. or kg/m) perpendicular to the generator (Fig. 5), and of dimension $2r\phi_1$, the load factor p_{rmni} is given by

$$P_{rmoi} = \frac{4\phi_1 q_i}{\pi l} \sin \frac{m\pi b_i}{l}, \quad m = 1, 2, 3, \ldots, \tag{7a}$$

and

$$p_{rmni} = \frac{8q_i}{\pi n l} \sin n\phi_1 \cos n\delta_i \sin \frac{m\pi b_i}{l}, \quad n, m = 1, 2, 3, \ldots, \tag{7b}$$

where b_i is equal to x_0 and x_f for $i = N_1 + 3$ to $N_1 + N_2 + 2$, and for $i = N_1 + N_2 + 3$ to $N_1 + 2N_2 + 2$, respectively (Fig. 2).

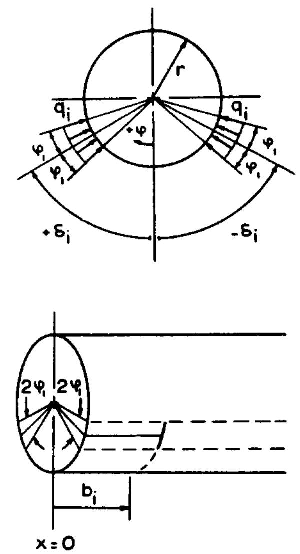

Fig. 5. Line load perpendicular to the generator, q_i (lb/in or kg/m).

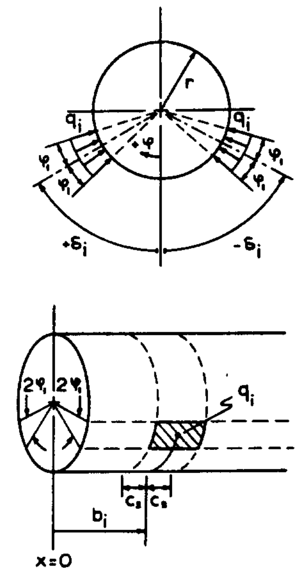

Fig. 6. Distributed loads, q_i (lb/in^2 or kg/m^2).

(d) *Constant pressure uniformly distributed over two rectangular areas.* Consider two rectangular areas subjected to a constant loading q_i, centered at the coordinates $(x, \phi) = (b_i, \pm\delta_i)$, and having the dimensions $2C_2$ and $2r\phi_1$ along the axial and circumferential directions, respectively (Fig. 6).

The Fourier series expansion of the pressure, p_{ri}, due to the radial loading, q_i, being given by eqn (2), one obtains the following expression for the load factor p_{rmni}

$$p_{rmoi} = \frac{8\phi_1 q_i}{\pi^2 m} \sin\frac{m\pi C_2}{l} \sin\frac{m\pi b_i}{l}, \quad \text{for} \quad m = 1, 2, 3, \ldots, \tag{8a}$$

and

$$p_{rmni} = \frac{16 q_i}{\pi^2 mn} \sin n\phi_1 \sin\frac{m\pi C_2}{l} \cos n\delta_i \sin\frac{m\pi b_i}{l}, \quad \text{for} \quad m, n = 1, 2, 3, \ldots, \tag{8b}$$

where $i = N_1 + 2N_2 + 3, \ldots, N$; and N is given by eqn (1).

3.2 *Particular cases of arbitrary loading*

When the loading can be represented by Fourier series of the form

$$\begin{Bmatrix} p_x \\ p_r \\ p_\phi \end{Bmatrix} = \sum_{n=0,2,3,4,\ldots}^{\infty} [T_n] \begin{Bmatrix} p_{xon} \\ p_{ron} \\ p_{\phi on} \end{Bmatrix}. \tag{9}$$

The displacements can, in general, be assumed to have the form

$$\begin{Bmatrix} U_p \\ W_p \\ V_p \end{Bmatrix} = \sum_{n=0,2,3,4,\ldots}^{\infty} [T_n] \begin{Bmatrix} u_{on} \\ w_{on} \\ v_{on} \end{Bmatrix} \tag{10}$$

where the coefficients u_{on}, w_{on} and v_{on} for $n \neq 1$, may be obtained from eqn (4) by imposing $m = 0$.

(a) *Pressure corresponding to a partially-filled vessel.* The pressure distribution on a shell partially or completely filled with stationary liquid is shown in Fig. 7. The series expansion for such

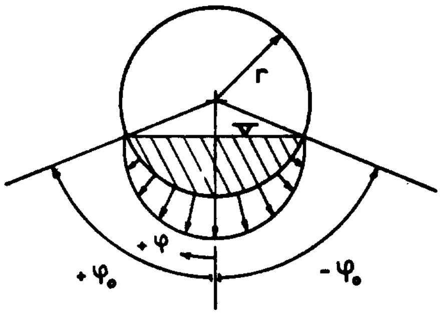

Fig. 7. Pressure distribution for a partially-filled shell.

loading is given by eqn (9) and its corresponding load factor may be written in the form

$$p_{ro} = -(\gamma r/\pi)[\sin\phi_o - \phi_o\cos\phi_o],$$
$$p_H = -(\gamma r/\pi)[\phi_o - (\sin 2\phi_o/2)],$$
$$p_m = -[2\gamma r/(\pi n(n^2-1))][\cos\phi_o\sin n\phi_o - n\sin\phi_o\cos n\phi_o],\quad n = 2, 3, \ldots, \tag{11}$$

where γ (lb/in^3 or kg/m^3) is the specific weight of the fluid and ϕ_o (radians) indicate the level of the liquid in the shell.

(b) *Surcharge pressure, weight of vessel and heads.* The loading corresponding to a surcharge pressure p_o is given by

$$p_r(x, \phi) = -p_o, \tag{12}$$

and the loading due to the weight of the vessel may be written as

$$p_\phi(x, \phi) = -\gamma_s t\sin\phi, \tag{13}$$

$$p_r(x, \phi) = -\gamma_s t\cos\phi, \tag{14}$$

where γ_s is the specific weight of the shell's material and t the thickness of the vessel.

The heads of the vessel are assumed to be rigid and, consequently, their effects will be taken into account by prescribing the appropriate boundary conditions on the shell's edges.

3.3 *Stresses and displacements*

The displacements due to arbitrary distributed loadings may be obtained by introducing the coefficients given by eqns (5)–(8) and (11)–(14) into relations (4). However, eqns (4) and (10) cannot be applied to cases where the loadings are expressed in terms of the form ($a\cos\phi$) and ($b\sin\phi$); this is due to the fact that expressions in ($d\cos\phi$) and ($g\sin\phi$), for the radial displacement W, correspond to rigid body motion of the shell and therefore, they do not represent the true displacement caused by loadings expressed in terms of the same form. In order to avoid this difficulty, it was necessary to either expand the constant "a" and "b" in a Fourier sine series of the form $\Sigma D_n \sin(m\pi x/l)$ and thus to lengthen the numerical computations, or to obtain a particular solution using shells' membrane theory. It was decided to use the latter alternative, for these particular case since such solution which is easily obtainable describes adequately the behavior of a cylindrical shell closed by stiff heads. Such special cases occur when $m = 0$ and $n = 1$, e.g. for the weight of the shell and for the terms, $m = 0$ and $n = 1$, of the fluid pressure. In all other cases, where ($m = 0$, $n \neq 1$) and $m \neq 0$, the exact solution, developed in previous sections, was used with bending resistance under arbitrary loads to determine the corresponding displacements and stresses.

The stress-resultant vector, $\{\sigma\} = \{N_x, N_\phi, \bar{N}_{x\phi}, M_x, M_\phi, M_{x\phi}\}^T$, for different loadings is obtained by substituting the corresponding displacement relations into eqn (35) and (36) of Ref. [9].

(c) *Surcharge pressure.* The surcharge pressure p_o induces displacements and stresses for the case $n = 0$; accordingly, we may write

$$\begin{Bmatrix} U \\ W \\ \partial W/\partial x \\ V \end{Bmatrix}_{p_o} = \begin{Bmatrix} p_o rx(1-2\nu)/2Et \\ p_o r^2(1-0.5\nu)/Et \\ 0 \\ 0 \end{Bmatrix}, \tag{15}$$

and

$$\{\sigma\}_{po} = \{p_o r/2, p_o r, 0, 0, 0, 0\}^T. \tag{16}$$

(b) *Weight of the shell.* The displacement and stress-resultant vectors of the weight of the shell which induces motions in the first circumferential wavenumber ($n = 1$) are given, respectively, by eqns (22) and (23) of Ref. [7].

(c) *Fluid pressure.* Upon substituting relations (11) into eqns (4), (10) and thence into eqns (36) of Ref. [9] we obtain the following expressions for the displacements and the stress-resultants of the shell:

$$\begin{Bmatrix} U \\ W \\ \partial W/\partial x \\ V \end{Bmatrix}_{p_f} = \begin{Bmatrix} U \\ W \\ \partial W/\partial x \\ V \end{Bmatrix}_{p_f(n=0)} + \begin{Bmatrix} U \\ W \\ \partial W/\partial x \\ V \end{Bmatrix}_{p_f(n=1)} + \sum_{n=2}^{\infty} [\tau_n] \begin{Bmatrix} u_{on} \\ w_{on} \\ 0 \\ v_{on} \end{Bmatrix}_{p_f}, \tag{17}$$

and

$$\{\sigma\}_{p_f} = \{\sigma\}_{p_f(n=0)} + \{\sigma\}_{p_f(n=1)} + \sum_{n=2}^{\infty} \begin{bmatrix} T_n & 0 \\ 0 & T_n \end{bmatrix} [E_{on}] \begin{Bmatrix} u_{on} \\ w_{on} \\ v_{on} \end{Bmatrix}, \tag{18}$$

where $[U, W, \partial W/\partial x, V]^T_{p_f(n=0)}$ and $[U, W, \partial W/\partial x, V]^T_{p_f(n=1)}$ are given by eqns (26) and (27) of Ref. [7], respectively; and

$$\{\sigma\}_{p_f(n=0)} = \{-a_o r/2, -a_o r, 0, 0, 0, 0\}^T, \tag{19}$$

$$\{\sigma\}_{p_f(n=1)} = \left\{a_1 \left[\frac{l^2}{2r}\left(\frac{x^2}{l^2} - \frac{x}{l}\right) - \frac{r}{4}\right] \cos\phi, -ra_1\cos\phi, a_1\left(\frac{l}{2} - x\right)\sin\phi, 0, 0, 0\right\}^T, \tag{20}$$

a_o and a_1 are, respectively, equal to p_{ro} and p_{r1} of eqns (11); the matrices $[T_n]$, $[\tau_n]$ and $[E_{on}]$ are shown in the Appendix; and the vector $\{u_{on}, w_{on}, v_{on}\}^T_{p_f(n\geq 2)}$ is determined by substituting relations (11) into eqns (4).

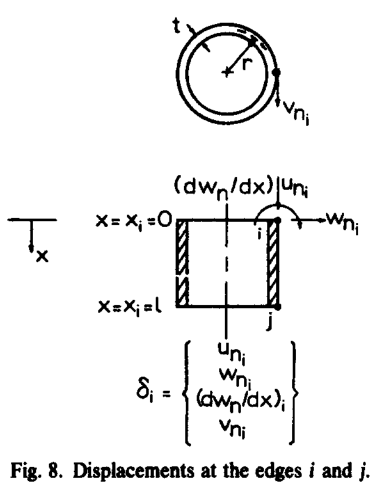

Fig. 8. Displacements at the edges i and j.

(d) *Point, line and surface loads.* Similarly, the stresses and displacements due to the applied concentrated loads, line loads and surface loads (q_i, $i = 1, N$) (Fig. 2), may be written as follows

$$\begin{Bmatrix} U \\ W \\ \partial W/\partial x \\ V \end{Bmatrix}_{q_i} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} [\tau_n][\tau_{mx}] \begin{Bmatrix} \sum_{i=1}^{N} u_{mni} \\ \sum_i w_{mni} \\ \sum_i (m\pi/l) w_{mni} \\ \sum_i v_{mni} \end{Bmatrix}_{q_i} \tag{21}$$

and

$$\{\sigma\}_{q_i} = \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} \begin{bmatrix} T_n & 0 \\ 0 & T_n \end{bmatrix} \begin{bmatrix} X_m & 0 \\ 0 & X_m \end{bmatrix} [P][C_{mn}] \begin{Bmatrix} \sum_{i=1}^{N} u_{mni} \\ \sum_i w_{mni} \\ \sum_i v_{mni} \end{Bmatrix}_{q_i}, \tag{22}$$

where $[P]$, the elasticity matrix, is given in Ref. [9]; the matrices $[\tau_n]$, $[\tau_{mx}]$, $[T_n]$, $[X_m]$ and $[C_{mn}]$ are given in the Appendix; and the vector $\sum_{i=1}^{N} \{u_{mni}, w_{mni}, v_{mni}\}_{q_i}^T$ is evaluated by substituting the relations (5)–(8) corresponding, respectively, to $i = 1$ to 2, 3 to N_1+2, N_1+3 to N_1+2N_2+2, and N_1+2N_2+3 to N, into eqns (4); Fig. 2.

4. BOUNDARY CONDITIONS

In this development we take into account the effects of the boundary conditions. The solution for arbitrary edge loading is given in Ref. [9], and only an outline will be given here for the benefit of the reader.

We consider motions in the nth circumferential wavenumber and write

$$\begin{Bmatrix} U(x,\phi) \\ W(x,\phi) \\ V(x,\phi) \end{Bmatrix}_c = \sum_{n=0}^{\infty} [T_n] \begin{Bmatrix} u_n(w) \\ w_n(x) \\ v_n(x) \end{Bmatrix}, \tag{23}$$

where $[T_n]$ is given in the Appendix. Substituting this into Sander's equations of equilibrium [9] by assuming $p_x = p_\phi = p_r = 0$, and letting

$$u_n(x) = A\,e^{\lambda x/r},\ v_n(x) = B\,e^{\lambda x/r},\ w_n(x) = C\,e^{\lambda x/r} \tag{24}$$

leads to a final expression of the displacement vector at the boundaries as follows

$$\begin{Bmatrix} U \\ W \\ \partial W/\partial x \\ V \end{Bmatrix}_C = \sum_{n=0}^{\infty} [\tau_n][AX]\{C_n\}, \tag{25}$$

where the matrices $[\tau_n]$ and [AX] are shown in the Appendix, and $\{C_n\} = \{\bar{C}_1, \ldots, \bar{C}_8\}^T$ is a set of eight constants. The corresponding stress-resultant vector $\{\sigma\}_C$ is given by eqn (5) of Ref. [7]. The $\bar{C}_j$ where $j = 1, \ldots, 8$, are the only free constants in our problem and must be determined from eight boundary conditions, four at each edge of constant x.

Using the principle of superposition, it is now possible to determine the total boundary forces and displacements for each specific n (Fig. 8):

(a) for axisymmetric loads where $n = 0$, we have

$$\begin{Bmatrix} \delta_{oi} \\ \delta_{oj} \\ F_{\delta oi} \\ F_{\delta oi} \end{Bmatrix} = \begin{bmatrix} [A] \\ [AF_i] \\ [AF_j] \end{bmatrix} \{C_o\} + \{\bar{p}_o\} + (\bar{p}_{fo}\} + \{\bar{q}_o\}, \tag{26}$$

(b) for the case of the beam-like mode ($n = 1$), we obtain

$$\begin{Bmatrix} \delta_{1i} \\ \delta_{1j} \\ F_{\delta 1i} \\ F_{\delta 1j} \end{Bmatrix} = \begin{bmatrix} [A] \\ [AF_i] \\ [AF_j] \end{bmatrix} \{C_1\} + \{\bar{p}_1\} + \{\bar{p}_{f1}\} + \{\bar{q}_1\}, \tag{27}$$

and finally the prescribed boundary forces and displacements for non-axisymmetric loads ($n \geqslant 2$) are defined by

$$\begin{Bmatrix} \delta_{ni} \\ \delta_{nj} \\ F_{\delta ni} \\ F_{\delta nj} \end{Bmatrix} = \begin{bmatrix} [A] \\ [AF_i] \\ [AF_j] \end{bmatrix} \{C_n\} + \{\bar{p}_{fn}\} + \{\bar{q}_n\},$$

where the matrices $[AF_i]$, $[AF_j]$, $\{\bar{p}_o\}$, $\{\bar{p}_{fo}\}$, $\{\bar{q}_o\}$, $\{\bar{p}_1\}$, $\{\bar{p}_{f1}\}$, $\{\bar{q}_1\}$, $\{\bar{p}_{fn}\}$ and $\{\bar{q}_n\}$ are given in the Appendix. The vector $[\delta_{ni}, \delta_{nj}, F_{\delta ni}, F_{\delta nj}]^T$ is given by eqns (34) and (35) of Ref. [7].

In eqns (26)–(28), $[A]$ is an 8×8 matrix obtained from $[AX]$ which is a 4×8 matrix of eqn (25): the upper part of $[A]$ corresponds to $x = 0$ and the lower one refers to $x = l$. The (4×8) matrices $[AF_i]_{x=0}$ and $[AF_j]_{x=l}$ are calculated by relations (33) and (35) of Ref. [7]. The vectors $\{\bar{p}_o\}$, $\{\bar{p}_1\}$, $\{\bar{p}_{fn}\}$ and $\{\bar{q}_n\}$ where $n = 0, 1, 2, \ldots$, are the boundary displacements and forces corresponding respectively, to the surcharge pressure, shell's weight, fluid loads, and point, line and surface loads. Eight of the sixteen components of the vector $\{\delta_{ni}, \delta_{nj}, F_{\delta ni}, F_{\delta nj}\}^T$, $n = 0, 1, 2, \ldots$, represent the total number of support conditions that are to be considered on the edges of the structure.

We now have the stresses and displacements of a shell subjected to external and arbitrary edge loadings given in terms of the unknowns q_i, $i = 1, 2, \ldots, N$, by eqns (15)–(22) with the edge constants $\{C_n\}$ given by eqns (26)–(28) and the other terms involved given by eqns (5)–(14).

5. FORMULATION OF THE CONTACT PROBLEM

Had the pressure distribution, q_i, $i = 1, \ldots, N$, been known, the response as expressed by eqns (15)–(24) would have been the solution to the problem. In the case of unknown q_i's, however, we must proceed differently in order to obtain the pressure distribution and contact area when the applied loads and the supports' configurations are prescribed. In this section we shall establish the conditions for overall equilibrium of the system and express the total potential energy in terms of the q_i's, vessel self weight, surcharge pressure, fluid and edge loadings.

These q_i's may then be obtained by minimizing the quadratic form of the potential energy of the system with the necessary constraints to insure proper contact with the supports. Such minimisation is executed using SUMT method (Sequential Unconstrained Minimum Technique) [12]. This technique is briefly explained in Section 5.3.

5.1 *Contact criteria*

We consider rigid supports and a cylindrical shell governed by the laws of linear elasticity. As previously stated, the area between the supports and the shell is subdivided into N elements, each of which subjected to an unknown pressure distribution q_i where $i = 1, 2, \ldots, N$. The corresponding forces F_{qi} acting at the discrete points ($i = 1, \ldots, N$) are as follows (see Figs. 3–6).

$$\begin{aligned} F_{q_i} &= q_i,\ i = 1, 2 \\ F_{q_i} &= 2C_2 q_i,\ i = 3, 4, \ldots, N_1 + 2, \\ F_{q_i} &= 2\phi_1 r q_i,\ i = N_1 + 3, \ldots, N_1 + 2N_2 + 2, \end{aligned} \tag{29}$$

and

$$F_{q_i} = 4\phi_1 C_2 r q_i,\ i = N_1 + 2N_2 + 3, \ldots, N, \tag{30}$$

where N is given by eqn (1).

The normal displacement w at any point i in the proposed zone of contact must be smaller or equal to zero (assuming no rigid body motion of the shell). On the other hand, the sum of all vertical components of the forces acting at the discrete points ($i = 1, \ldots, N$) must balance the total weight of the system. This equilibrium condition can therefore be written as

$$\sum_{i=1}^{N} F_{q_i} \cos \delta_i = P, \tag{31}$$

where P is the total weight of the system and the δ_i's are shown in Figs. 3–6. The criterion for contact may be given by the following constraints:

If

$$W_{qi} < 0, \quad \text{then} \quad F_{q_i} = 0, \tag{32}$$

and if

$$W_{q_i} = 0, \quad \text{then} \quad F_{q_i} \geq 0, \tag{33}$$

where W_{q_i} is radial elastic displacement at point i; and the eqns (32) and (33) represent the "no contact" and "contact" regions, respectively.

5.2 *Potential energy and its minimization*

The total potential energy, $\pi(q_i)$, of the system may be expressed in terms of the strain energy, U_S, and the work of the external and edge forces W_{EX} and W_{BC}, respectively.

The determination of U_S, W_{EX} and W_{BC} is carried out in Ref. [7].

The general expression of the potential energy is given by

$$\pi(q_i) = \frac{1}{2} U_S - W_{EX} - W_{BC}, \tag{34}$$

where U_S, W_{EX} and W_{BC} are given by eqns (44)–(46) of Ref. [7], respectively.

The original contact problem may then be stated as follows:

Minimize

$$\left.\begin{array}{l} \pi(q_i), \\ \text{such that} \quad \displaystyle\sum_{i=1}^{N} F_{q_i} \cos \delta_i = P, \end{array}\right\} \tag{35}$$

and subject to the condition that either $W_{q_i} = 0$ or $F_{q_i} = 0$; $F_{qi} \geq 0$, $W_{q_i} \geq 0$; where π, F_{q_i} and P are given by eqns (34), (29) and (31), respectively.

Equation (35) is minimized by using SUMT method[12] in order to determine the pressure distribution.

5.3 *SUMT method, a penalty function technique*

This method is based on transforming a given constrained minimization problem into a sequence of unconstrained minimization problems. This transformation is accomplished by defining an appropriate auxiliary function, whose minima are unconstrained in some domain in terms of the objective and constraint functions. By gradually removing the effect of the constraints in the auxiliary function by controlled charges in the value of a parameter, a family of unconstrained problems is generated that have solutions converging to a solution of the original constrained problem.

The auxiliary function used in this case is a logarithmic penalty function having the form

$$P(q_i, r_k) = \pi(q_i) - r_k \sum_{i=1}^{m} \ln gi(q_i) + \sum_{i=m+1}^{n} \frac{\{\min [0, gi(q_i)]\}^2}{r_k} \tag{36}$$

where $gi(q_i)$ is equal to W_{qi} or F_{qi}.

The computational algorithm is summarized as follows[12]:

1. Find a point q_o in $R_o = \{q_i | gi(q_i) > 0, i = 1, \ldots, m\}$. If such a point is not available the optimization method may be used to find it.
2. Determine r_1, the initial value of r_k.
3. Determine the unconstrained minimum of $P(q_i, r_k)$ for the current value of r_k. "This step constitutes the bulk of the work required by unconstrained algorithms".
4. Extimate the solution of the problem using the extrapolation theory of [12].
5. Terminate computations if the estimated solution is "acceptable". A natural criterion for convex programming problems is available via the theory of duality.
6. Select r_{k+1}.
7. Estimate the next unconstrained minimum of the $P(q_i, r_k)$ function using the extrapolation formula. Continue from Step 3.

6. CALCULATIONS AND DISCUSSION

To determine the interface pressure distribution q_i's, the displacements and the stresses of a given cylindrical vessel completely or partially filled with liqiuid, pressurized and having a surface of contact with rigid supports, we first specify the imposed boundary conditions, their number J, the location and dimensions of the supports. The surfaces of these supports are then subdivided into a sufficient number, N, of line and surface elements each of which subjected to a load q_i where $i = 1, 2, \ldots, N$ (sufficiency in this context is related to the required degree of precision). A computer programme, written in Fortran V language forthe CDC CYBER 74 computer, determines all the q_i's by minimizing, for given input data, the potential energy of the system, calculates the displacements and stresses for each particular loading, obtains the values of the constrants $\{C_n\}$ corresponding to the appropriate boundary conditions and finally determines the total response at any point of the structure.

The additional necessary input data are the mean radius r, wall thickness t, length of the vessel l, specific weights of material of the shell and of the fluid γ_X and γ, respectively; Poisson's ratio ν, modulus of elasticity E, internal pressure p_o and the angle ϕ_o (rad.) which represents the level of the liquid in the shell.

The computer programme proceeds as follows:

(1) The eight complex roots, λj, of the characteristic equation given in Appendix 2 of Ref. [7], are calculated, for each circumferential wavenumber ($n \geq 0$), by the Newton–Raphson iterative technique, and hence, we obtain the parameters κ_1, κ_2, μ_1, μ_2, α_p, β_p and $\bar{\alpha}_p \cdot \bar{\beta}_p$ ($p = 1, 2, \ldots, 8$) shown in the matrices $[Q]$, $[AX]$, $[AF_i]$ and $[AF_j]$.

(2) The displacements and stress-resultant matrices corresponding to the surcharge pressure, shell's weight and fluid pressure are computed, respectively, by the relations given by eqns (15)–(16), (22)–(23) of [7], and (17)–(20).

(3) The total potential energy of the system, $\pi(q_i)$, given by eqn (34) in terms of the unknown q_i's, $i = 1, \ldots, N$, and subjected to the conditions given by relation (35) is minimized by using SUMT method[12] in order to determine the pressure distribution. Then, the stresses and displacements due to the q_i's are computed by the relationships given by eqns (21) and (22).

(4) If the boundary conditions of the shell are under consideration, then appropriate rows of relations (26)–(28) are deleted to satisfy these conditions, reducing the matrix equations to one of order J where J is the number of boundary conditions imposed. Thus for non-axisymmetrical loads, four boundary conditions will have to be prescribed at each edge and $J = 8$; and for axisymmetric loads ($n = 0$) only two boundary conditions will be required at each edge and $J = 4$. With the reduced eqns (26)–(28) and its intermediate matrices determined, the computer program proceeds to find the eight constants of the vector $\{C_n\}$; and consequently, the stress-resultants and displacements caused by the imposed boundary conditions and given, respectively, by eqns (5) of [7] and (25) of this paper may be determined at any point of the shell.

The total number of n required for the computation of the total stress-resultant and displacements is reached when the relative error of each displacement components approaches 10^{-9}.

The necessary time for the minimization of the potential energy in order to obtain the q_i's seems to be high. However if only a few elements are used in the calculation, the response may be computed to an acceptable degree of accuracy, but with saving in computational cost.

Of course, as the answer to any particular problem has to be obtained numerically, the proof of usefulness of this theory, as compared to other theories, will also depend on its efficiency (in terms of cost and effort), as well as its precision. A paper currently under preparation will deal with the numerical minimization of the potential energy, its limitations and usefulness as well as the necessary simplifications to avoid high computational cost. Two categories of contact problem will be dealt with: (a) the evaluation of the contact area and the pressure distribution when the applied loads are of the contact area and the pressure distribution when the applied loads are known and (b) the design of systems capable of giving the best possible distribution of pressure over the contracting regions.

Here only two typical cases has been calculated in order to check the correctness of the theory. The first set of calculations was undertaken to determine the displacements and stresses of a particular twin saddle supported vessel partially filled with liquid. The shell analysed is one already studied by Forbes and Tooth[5], with whose results those of this theory will be compared. The vessel, manufactured from an aluminium alloy sheet, is subjected at various water levels to self weight and interface pressure. The resulting interace pressure distribution between the saddle and the vessel is shown in Fig. 9 as given by Ref. [5]. The data of the simply supported vessel are as follows: $r = 5.625$ in (14.287 cm), $t = 0.028$ in (0.07112 cm), $l = 53$ in (1.3462 cm), $E = 10^7$ lbf/in^2 (0.703103×10^{10} kg/m^2), $\nu = 0.3$, $\gamma = 0.03611$ lbf/in^3 (999.52 kg/m^3), $\gamma_S = 0.09754$ lbf/in^3 (2.7×10^3 kg/m^3). The saddles are maintained at a constant distance ($11l/60$) from the vessel's ends.

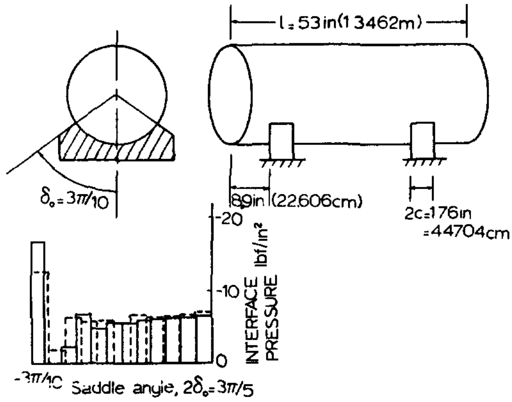

Fig. 9. Analytical and experimental saddle/cylinder interface pressure of [5].

In the experiments of [5] the liquid depth was varied such that the angle which represent the level of the liquid in the shell took the values $\phi_o = 0.4644\pi$, 0.60833π, 0.78333π and π radians. The effects of the closed ends were taken into account. For each ϕ_o, the radial and circumferential displacements were measured by Forbes and Tooth at the saddle outer rim profile and at the vessel center profile for a number of values of the circumferential coordinate, ϕ. Forbes and Tooth also developed a method based on Flügge's theory which, however, only applies when the shell is empty or completely full ($\phi_o = \pi$). In their investigation, the distribution and magnitude of the interface pressure is assumed to be (1) the same for both saddles, (2) symmeteic with respect to the generator passing through the center of the saddle arc and (3) constant across the saddle width. Finally the saddle arc length is subdivided into a series of equal angular parts each of axial length equal to the width of the saddle and loaded by a uniform pressure.

The stresses and displacements were calculated by our theory using 20 and 12 elements in the axial and circumferential directions, respectively, for a total number of 286 elements (see eqn 1) in the case of $\phi_o = 0.4644\pi$, 0.60833π, 0.78333π and π radians. In all cases the finite elements were of equal length and the value of n_{max} required for reaching a relative error of 10^{-9} is equal to 24. The necessary CPU time for the calculation of this case is approximately 25 min. Figures 10–15 show our computed results compared with the experimental data of [5]. Agreement between theory and experiment is quite good in most cases.

One noteworthy observation is that, generally, the theory somewhat overestimates the radial displacements, v, at all fractional fillings, while it estimates them almost accuately when the shell is

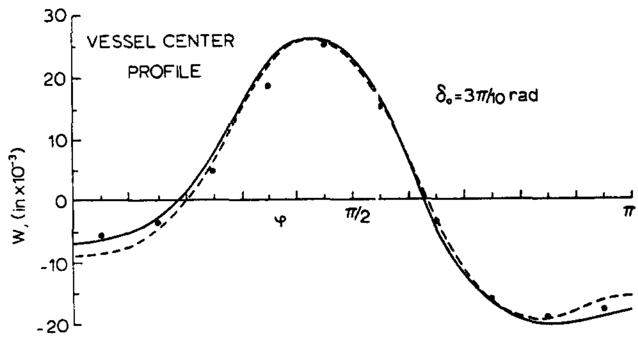

Fig. 10. Radial displacements of a twin saddle supported cylindrical vessel full of liquid. —, this theory; --- theory of Forbes and Tooth[5]; ●, experimental results of [5].

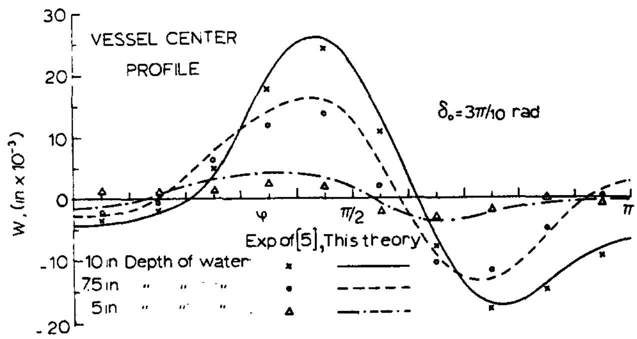

Fig. 11. Radial displacements of a twin saddle supported cylindrical vessel at various water levels.

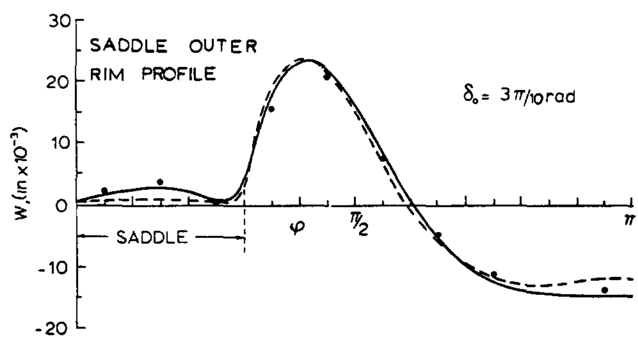

Fig. 12. Radial displacements at saddle outer rim profile of saddle supported cylindrical vessel full of liquid; ——, this theory; ---, theory of [5]; ●, experimental results of [5].

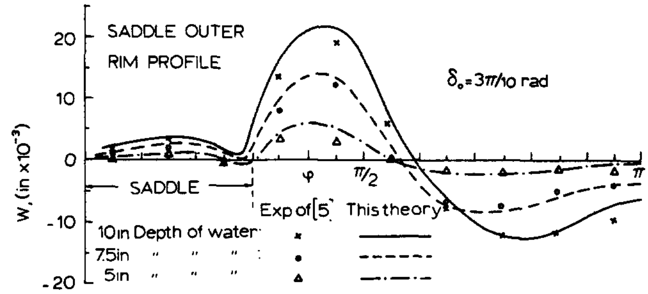

Fig. 13. Radial displacements at various water levels.

completely filled. A possible, reasonable explanation is that the shell is not ideally simply-supported assuming of course that the experimental values are correct.

The second set of calculations involves the determination of the displacement and stresses of a particular twin saddle supported vessel which has been analysed by Wilson and Tooth[6]. The

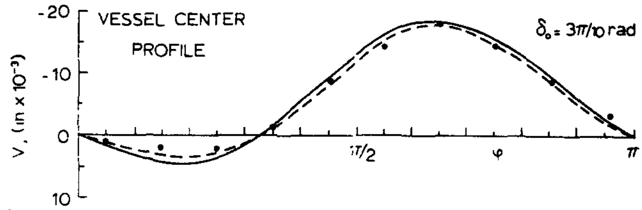

Fig. 14. Circumferential displacements of saddle-supported cylindrical vessel full of liquid. ——, this theory; ---, theory of [5]; ●, experimental results of [5].

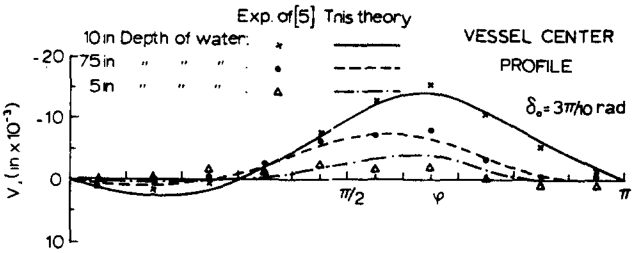

Fig. 15. Circumferential displacements at various water levels.

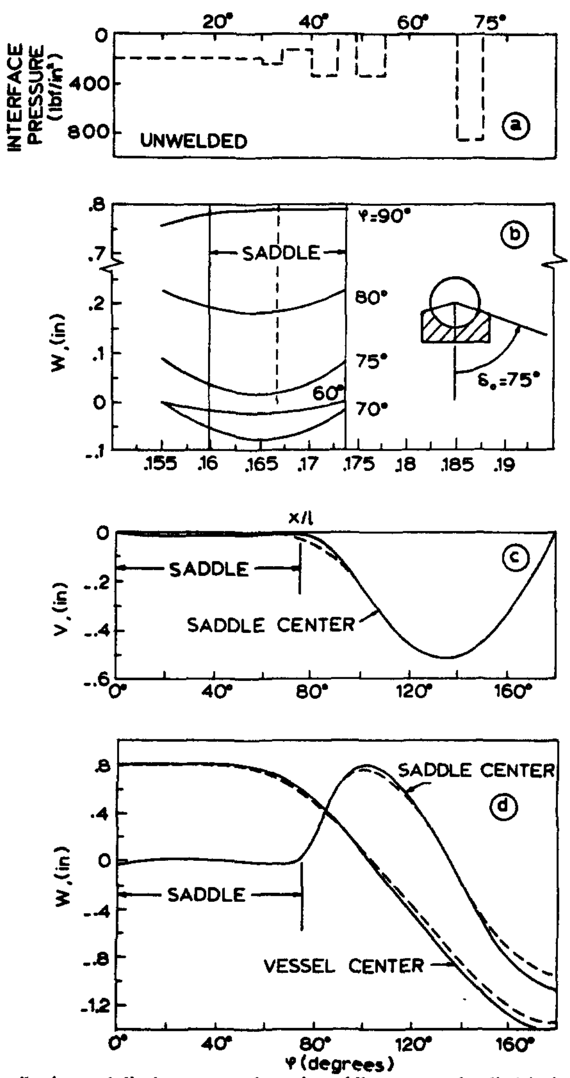

Fig. 16. Pressure distribution and displacements of a twin saddle supported cylindrical vessel full of liquid. ——, this theory; --- theory of Wilson and Tooth [6].

vessel is subjected to water loading, self weight and interface pressure. The resulting interface pressure distribution between the saddle and the vessel is shown in Fig. 16(a) as given by Ref. [6]. The data for the vessel are as follows: $r = 6$ ft (1.85 m), $t = 1$ in (25.4 mm), $l = 180$ ft (54.9 m), $E = 0.29 \times 10^8$ lb/in^2 (0.2039×10^{11} kg/m^2), $\nu = 0.3$, $\gamma = 0.03611$ lb/in^3 (999.52 kg/m^3), $\gamma_s = 0.284$ lb/in^3 (7.8×10^3 kg/m^3) and $\phi_o = \pi$ rad. The location of the twin saddles on the vessel is shown in Fig. 17(a). The effect of the closed ends is taken into account. The results obtained by the present theory were computed with $N_1 = 49$, $N_2 = 15$ and $N = 816$ elements, and are compared with those of [6] in Fig. 16–18. As may be seen, the results obtained by this theory are generally in quite good agreement with those of [6].

The results for the radial displacement, W, in terms of the saddle's width are shown in Fig. 16(b). We note that these displacements are not constant along the width of the saddle for ϕ smaller than the saddle angle δ_o, ($\phi \le 75°$), but tend to a uniform value for ϕ higher than δ_o. Therefore, our results show either an incompatibility regarding the assumption, used in [6], that the interface pressure is constant across the saddle width or the saddle is not rigid in the axial direction.

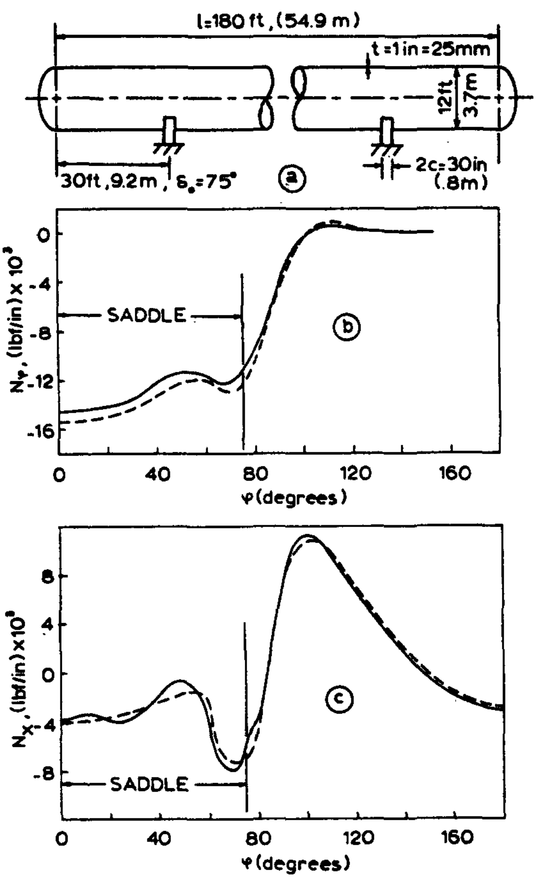

Fig. 17. Values of N_φ and N_x at saddle center profil for fluid and self weight. ——, this theory; ---, theory of [6].

CONCLUSION

As developed previously, the present theory and the computer program upon it, is capable of determining the stresses, displacements and the interface pressure distribution of the general case of a thin cylindrical shell with arbitrary boundary conditions, pressurized, partially or completely filled with liquid and having two or more supports. To this end the supports are subdivided, in the circumferential and axial directions, into a sufficient number, N, of line and surface elements each of which is subjected to a load q_i, $i = 1, \ldots, N$. SUMT method is used to minimize the potential energy in order to determine those q_i's.

This theory was computerized so that if the dimensions and material properties of the

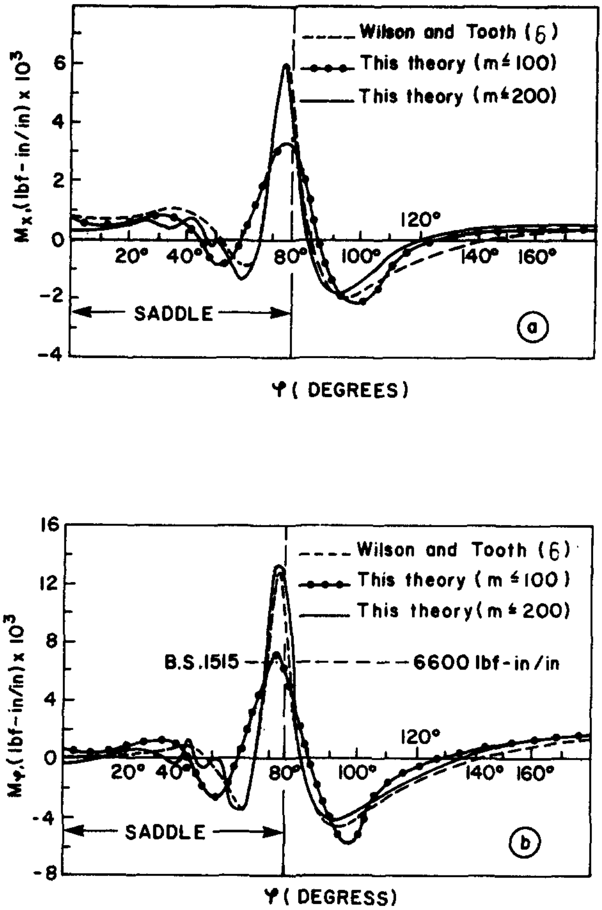

Fig. 18. Values of M_x and M_ϕ at saddle center profile for fluid and self weight.

vessel, and the properties of the saddle, are given as inputs, the program gives as output the displacements and stresses at any point of the structure.

As stated previously, further computations are under way to test the limitations and the usefulness of the minimization of the potential energy and to design a system capable of giving the best possible distribution of pressure over the contacting regions. Such development is the subject of another paper currently under preparation.

A number of cases, the authors believe, could have been tackled to illustrate the capabilities of the theory but were not because of the computational cost. Thus, shells with several discontinuities in loading could be analyzed with the same ease as those presented here. Similarly, cases of anisotropic shells may be analysed equally easily [13].

Acknowledgements—This research is supported by the National Research Council of Canada (Grants No. A8814 and C1346) whose assistance is hereby gratefully acknowledged.

REFERENCES

1. R. D. Stokes and G. Sved, Stresses in steel pipelines at saddle supports. *The Institution of Engineers, Australia, Civil Eng. Transaction*, 119–130 (Oct. 1965).
2. W. M. Wilson and E. D. Olson, *Tests on Cylindrical Shells*. University Illinois Eng. Experimental Station, *Bull.* No. 331 (23 Sept. 1941).
3. L. P. Zick, Stresses in large horizontal cylindrical pressure vessels on two saddle supports. *J. A.S. Weld*, **30** (1956).
4. V. Krupka, An analysis for lug or saddle-supported cylindrical pressure vessels. *1st Int. Conf. Press. Vessel Tech.*, Delft (1969).
5. P. D. Forbes and A. S. Tooth, An analysis for twin saddle-supported unstiffened cylinderical vessels. *Conf. on Recent Advances in Stress Analysis*, 47–58. Royal Aeronautical Soc., London (March 1968).
6. J. D. Wilson and A. S. Tooth, The support of unstiffened cylindrical vessels, *2nd Int. Conf. on Pressure Vessel Technology, Part I: Design and Analysis*, pp. 67–83. San-Antonio, Texas (Oct. 1973).
7. A. A. Lakis and R. Doré, General method for analysing contact stresses on cylindrical vessels. *Tech. Rep.* No. EP-75-R-30, Ecole Polytechnique, Montreal, Canada.

8. J. L. Sanders. An improved first approximation theory for thin shells. NASA-TR-R-24 (1959).
9. A. A. Lakis and M. P. Paidoussis. Dynamic analysis of axially non-uniform, thin cylindrical shells. *J. Mech. Engng Sci.* **14**, 49–72 (1972).
10. E. Reissner, A new derivation of the equations for the deformation of elastic shells. *Am. J. Math.* **63**, 177 (1941).
11. S. Timoshenko and S. Woinowsky-Krieger. *Theory of Plates and Shells*. McGraw-Hill, New York (1959).
12. A. V. Fiacco and G. P. McCormick. *Nonlinear Programming: Sequential Unconstrained Minimization Techniques.* Wiley, New York (1968).
13. A. A. Lakis and R. Doré. Free vibration and response to pressure field of anisotropic thin cylindrical shells. *Proc. Noise Shock Vib. Conf.*, pp. 88–99. Monash University, Melbourne, Australia (22–25 May 1974).

APPENDIX

In this Appendix are given the matrices referred to in the text which were too large to be included therein. The matrices are listed as follows.

$[T_n], [T_{mx}], [\tau_n], [X_m], [\tau_{mx}]$	(see Table 1)
$[E_{on}], [C_{mn}], [A_F]$	(see Table 2)
$[AX]$	(see Table 3, Ref. [7])
$[AF_i]$	(see Table 4, Ref. [7])
$[AF_l]$	(see Table 5, Ref. [7])
$\{\bar{p}_o\}, \{\bar{p}_{fo}\}, \{\bar{p}_1\}, \{\bar{p}_{f1}\}$	(see Table 6, Ref. [7])
$\{\bar{p}_{fn}\}_{n\geq 2}, \{\bar{q}_n\}_{n\geq 0}, \{D_{on}\}_{n\geq 2}, \{S_{mn}\}_{\substack{m\geq 0\\ n\geq 0}}$	(see Table 7, Ref. [7])

Table 1. Matrices $[T_n]$, $[T_{mx}]$, $[\tau_n]$, $[X_m]$ and $[\tau_{mx}]$

$$[T_n] = \begin{bmatrix} \cos n\phi & 0 & 0 \\ 0 & \cos n\phi & 0 \\ 0 & 0 & \sin n\phi \end{bmatrix}$$

$$[T_{mx}] = \begin{bmatrix} \cos(m\pi x/\ell) & 0 & 0 \\ 0 & \sin(m\pi x/\ell) & 0 \\ 0 & 0 & \sin(m\pi x/\ell) \end{bmatrix}$$

$$[\tau_n] = \begin{bmatrix} \cos n\phi & 0 & 0 & 0 \\ 0 & \cos n\phi & 0 & 0 \\ 0 & 0 & \cos n\phi & 0 \\ 0 & 0 & 0 & \sin n\phi \end{bmatrix}$$

$$[X_m] = \begin{bmatrix} \sin(m\pi x/\ell) & 0 & 0 \\ 0 & \sin(m\pi x/\ell) & 0 \\ 0 & 0 & \cos(m\pi x/\ell) \end{bmatrix}$$

$$[\tau_{mx}] = \begin{bmatrix} \cos(m\pi x/\ell) & 0 & 0 & 0 \\ 0 & \sin(m\pi x/\ell) & 0 & 0 \\ 0 & 0 & \cos(m\pi x/\ell) & 0 \\ 0 & 0 & 0 & \sin(m\pi x/\ell) \end{bmatrix}$$

Table 2. Matrices $[E_{on}]$, $[C_{mn}]$ and $[A_F]$.

$$\left[E_{on}\right] = \begin{bmatrix} 0 & \nu D/r & \nu n D/r \\ 0 & D/r & nD/r \\ -nD(1-\nu)/2r & 0 & 0 \\ 0 & \nu n^2K/r^2 & \nu nK/r^2 \\ 0 & n^2K/r^2 & nK/r^2 \\ nK(1-\nu)/4r^2 & 0 & 0 \end{bmatrix}$$

$$\left[C_{mn}\right] = \begin{bmatrix} -m\pi/\ell & 0 & 0 \\ 0 & 1/r & n/r \\ -n/r & 0 & m\pi/\ell \\ 0 & (m\pi/\ell)^2 & 0 \\ 0 & n^2/r^2 & n/r^2 \\ n/2r^2 & (2n/r)\cdot(m\pi/\ell) & (3/2r)\cdot(m\pi/\ell) \end{bmatrix}$$

$$A_F(1,1) = r^2(m\pi/\ell)^2 + n^2\frac{(1-\nu)}{2}\left(1+\frac{k}{4}\right)$$

$$A_F(1,2) = -r\left(\frac{m\pi}{\ell}\right)\left[\nu - \frac{(1-\nu)}{2}\, k\, n^2\right]$$

$$A_F(1,3) = \frac{-rn}{2}\left(\frac{m\pi}{\ell}\right)\left[\left(1-\frac{3k}{4}\right) + \nu\left(1+\frac{3k}{4}\right)\right]$$

$$A_F 2,1) = A_F(1,2)$$

$$A_F(2,2) = 1 + kr^4\left[\frac{n^2}{r^2} + (m\pi/\ell)^2\right]^2$$

$$A_F(2,3) = n(1+n^2k) + \frac{(3-\nu)}{2}\, kr^2n\,(m\pi/\ell)^2$$

$$A_F(3,1) = A_F(1,3)$$

$$A_F(3,2) = A_F(2,3)$$

$$A_F(3,3) = \frac{(1-\nu)}{2}r^2\,(m\pi/\ell)^2\left(1+\frac{9k}{4}\right) + n^2(1+k)$$

The quantities ω_1, ω_2, η_1, η_2, ψ_1, ψ_2, ζ_1 and ζ_2 which appear in the matrices $[AX]$, $[AF_i]$ and $[AF_j]$ are given by $\omega_j = \kappa_j l/r$, $\eta_j = \mu_j l/r$, $\psi_j = \kappa_j x/r$, $\zeta_j = \mu_j x/r$; $j = 1, 2$, where κ_1, κ_2, μ_1, μ_2 are the real and imaginary components of the eight characteristic values λ_p, which may be written as [9],

$$\lambda_{1,2} = -\kappa_1 \pm \mu_1 i, \qquad \lambda_{3,4} = -\kappa_2 \pm \mu_2 i,$$

$$\lambda_{5,6} = \kappa_1 \pm \mu_1 i, \qquad \lambda_{7,8}\kappa_2 \pm \mu_2 i.$$

The quantities $\bar{\alpha}_p$, $\bar{\beta}_p$, $p = 1, 2, \ldots, 8$, are real or imaginary parts of the α_p and β_p defined in the paragraph following eqn (2), such that $\alpha_{1,2} = \bar{\alpha}_1 \pm \bar{\alpha}_2 i$, $\alpha_{3,4} = \bar{\alpha}_3 \pm \bar{\alpha}_4 i, \ldots, \bar{\alpha}_{7,8} = \bar{\alpha}_7 \pm \bar{\alpha}_8 i$; and similarly for the $\bar{\beta}_p$. The method for determining α_p, β_p is given in Ref. [9].